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A COMPARISON OF TWO METHODS OF LINEARIZED CHARACTERISTICS FOR A SIMPLE UNSTEADY FLOW

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SUMMARY

Two methods of using the concept of linearized characteristics are derived for the one-dimensional unsteady flow in a tube that is rotated about an axis perpendicular to the axis of the tube. One of the methods corresponds to that used by Ferri in his basic work on the subject. Solutions are made by both methods for boundary conditions that allow analytic solutions. Comparison shows that both methods give the same results but there are significant differences in their application.

INTRODUCTION

The term linearized characteristics has been applied by Ferri in reference 1 to a process of superposing a small perturbation on a basic solution of a set of nonlinear hyperbolic differential equations. The perturbation can be due to a change in the prescribed conditions along the boundary, in the position of the boundary, or in the differential equations themselves. The unknowns can each be expanded, for example, into a power series in a parameter representative of the perturbation, with unknown coefficients, except for the coefficient of the first term, representing the basic solution, which is known. The equations for the other coefficients are found to be linear and can be solved by characteristic methods. As applied to steady supersonic flow, this procedure has the advantage, stated roughly, over ordinary perturbation theory that it can be used for perturbations of any known flow (to which the characteristic method of solution can be applied) rather than only for perturbations of constant parallel flow.

A version of the process was used by Sauer (ref. 2) for determining the steady supersonic flow about bodies of revolution at small angles of attack but apparently has not been used for other problems except in reference 1, where its application to a variety of problems is discussed. In references 1 and 2, the expansion of the unknowns was substituted into the original equations of motion, which were then transformed into characteristic form. The purpose of the present paper is to compare that method of procedure (called method A subsequently) with an alternate method (method B) whereby expansions of the dependent and independent variables are substituted into the characteristic form of the equations.

The comparison is made by developing the equations of both methods as they apply to a simple set of nonlinear hyperbolic equations of gas dynamics, the set relating to the one-dimensional unsteady motion of a gas in a tube of constant cross section with the addition of a perturbation due to the rotation of the tube. The equations derived for this case are then solved for a set of boundary conditions for which the solution can be found analytically by both methods.

SYMBOLS

a,A	speed of sound
c_v	specific heat at constant volume
f,g	unknown functions of one variable
k	constant
p	pressure
R	gas constant
s,S	entropy increment divided by gas constant
t,T	time
u,U	velocity
x,X	distance along tube measured from axis of rotation
α,β	characteristic parameters
γ	ratio of specific heats
θ	absolute temperature

$\lambda = \frac{2c_v}{R} = \frac{2}{\gamma - 1}$

ρ density

ω angular velocity

Subscripts:

Letter subscripts indicate partial derivatives.

Number subscripts indicate order of approximation with the following exceptions:

a_0 reference value of a

x_0 and x_1 inner and outer ends of tube, respectively

ANALYSIS OF UNSTEADY ROTATING FLOW

Development of Equations

The linearized-characteristics concept is applied to a relatively simple problem in order to bring out more clearly some of its essential features. Consider the unsteady motion of an inviscid gas in a cylindrical tube which can rotate with angular velocity ω about an axis perpendicular to the generators of the cylinder (fig. 1). Distance along the tube is denoted by x measured from the axis of rotation. Only values of x large in comparison with the greatest lateral dimension of the tube in a plane of rotation are considered so that the local velocity of the rotation may be taken as ωx. Cross flows and lateral pressure gradients are neglected so that the flow considered is one-dimensional. A perfect gas with constant specific heats is assumed.

When $\omega = 0$, the ordinary equations of motion apply (ref. 3); that is, the continuity equation

$$\rho_t + \rho u_x + u\rho_x = 0 \tag{1}$$

and the momentum equation

$$u_t + uu_x = -\frac{1}{\rho} p_x \tag{2}$$

When $\omega \neq 0$, equation (1) is unaltered, but equation (2) has another term representing the acceleration along the tube due to the rotation

$$u_t + uu_x - \omega^2 x = -\frac{1}{\rho} p_x \tag{3}$$

The entropy is now assumed to be constant throughout the gas so that

$$p = k\rho^\gamma \tag{4}$$

See appendix A for a treatment of the case of variable entropy. Equation (4), the equation of state

$$p = \rho R\theta$$

and the relation

$$a^2 = \frac{dp}{d\rho} = \gamma R\theta$$

allow the replacement of p and ρ by functions of a in equations (1) and (3). Thus the equations of motion become

$$\left.\begin{aligned} \lambda a_t + au_x + \lambda ua_x &= 0 \\ u_t + uu_x + \lambda aa_x - \omega^2 x &= 0 \end{aligned}\right\} \tag{5}$$

where

$$\lambda = \frac{2c_v}{R} = \frac{2}{\gamma - 1}$$

These equations are referred to subsequently as the "primary equations" to distinguish them from the characteristic equations which are now developed.

Adding and subtracting equations (5) gives

$$\left.\begin{aligned} u_t + (u + a)u_x + \lambda\left[a_t + (u + a)a_x\right] - \omega^2 x &= 0 \\ u_t + (u - a)u_x - \lambda\left[a_t + (u - a)a_x\right] - \omega^2 x &= 0 \end{aligned}\right\} \tag{6}$$

By the usual procedure for hyperbolic equations (ref. 3), new independent variables α and β are introduced so that one equation of equations (6) has derivatives with respect to α only and the other, with respect to β only. Then the lines α = constant and β = constant form the two sets of characteristics. This process is accomplished by writing

$$\left.\begin{aligned} x_\alpha &= (u + a)t_\alpha \\ x_\beta &= (u - a)t_\beta \end{aligned}\right\} \tag{7}$$

so that

$$\frac{\partial}{\partial\alpha} = t_\alpha\left[\frac{\partial}{\partial t} + (u + a)\frac{\partial}{\partial x}\right]$$

$$\frac{\partial}{\partial\beta} = t_\beta\left[\frac{\partial}{\partial t} + (u - a)\frac{\partial}{\partial x}\right]$$

Thus, equations (6) become

$$\left.\begin{aligned} u_\alpha + \lambda a_\alpha - \omega^2 x t_\alpha &= 0 \\ u_\beta - \lambda a_\beta - \omega^2 x t_\beta &= 0 \end{aligned}\right\} \tag{8}$$

Equations (7) and (8) form a set of four equations in the four unknowns u, a, x, and t and are equivalent to equations (5). These equations can always be solved by step-by-step methods.

Method A.- Linearized approximations to equations (5) are now developed. Let

$$\left.\begin{aligned} u(x,t) &= U + \omega^2 u_1 + \omega^4 u_2 + \ldots \\ a(x,t) &= A + \omega^2 a_1 + \omega^4 a_2 + \ldots \end{aligned}\right\} \tag{9}$$

Only even powers of ω are used since equations (5) show that the problem is independent of the sign of ω. By substituting expressions (9) into equations (5) and setting the coefficients of each power of ω equal to zero, the following equations are obtained:

$$\left.\begin{aligned} \lambda A_t + AU_x + \lambda UA_x &= 0 \\ U_t + UU_x + \lambda AA_x &= 0 \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} \lambda a_{1t} + Au_{1x} + \lambda Ua_{1x} + U_x a_1 + \lambda A_x u_1 &= 0 \\ u_{1t} + Uu_{1x} + \lambda Aa_{1x} + U_x u_1 + \lambda A_x a_1 - x &= 0 \end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} \lambda a_{2t} + Au_{2x} + \lambda Ua_{2x} + U_x a_2 + \lambda A_x u_2 + a_1 u_{1x} + \lambda u_1 a_{1x} &= 0 \\ u_{2t} + Uu_{2x} + \lambda Aa_{2x} + U_x u_2 + \lambda A_x a_2 + u_1 u_{1x} + \lambda a_1 a_{1x} &= 0 \end{aligned}\right\} \tag{12}$$

and so forth. Or, when these equations are put into characteristic form by the same method that was applied to equations (5), the same equations for the characteristics are found to apply to each set; that is,

$$\left.\begin{aligned} x_\alpha &= (U + A)t_\alpha \\ x_\beta &= (U - A)t_\beta \end{aligned}\right\} \tag{13}$$

whereas the other equations are successively

$$\left.\begin{aligned} U_\alpha + \lambda A_\alpha &= 0 \\ U_\beta - \lambda A_\beta &= 0 \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} u_{1\alpha} + \lambda a_{1\alpha} + (U_x + \lambda A_x)t_\alpha(u_1 + a_1) - xt_\alpha &= 0 \\ u_{1\beta} - \lambda a_{1\beta} + (U_x - \lambda A_x)t_\beta(u_1 - a_1) - xt_\beta &= 0 \end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} u_{2\alpha} + \lambda a_{2\alpha} + (U_x + \lambda A_x)t_\alpha(u_2 + a_2) + (u_{1x} + \lambda a_{1x})t_\alpha(u_1 + a_1) &= 0 \\ u_{2\beta} - \lambda a_{2\beta} + (U_x - \lambda A_x)t_\beta(u_2 - a_2) + (u_{1x} - \lambda a_{1x})t_\beta(u_1 - a_1) &= 0 \end{aligned}\right\} \tag{16}$$

and so forth.

The development and use of equations (13) to (16) is referred to hereafter as "method A."

Method B.- Instead of substituting expansions of u and a into the primary equations (eqs. (5)) as in method A, these expansions are now substituted into the characteristic form of the equations (eqs. (8)). But the values u and a must be considered functions of α and β in equations (8), and the corresponding values of x and t are found from equations (7). Hence, x and t are also expanded. The method is, therefore, similar to the technique presented by Lighthill in references 4 and 5.

By writing

$$\left.\begin{aligned} u(\alpha,\beta) &= U + \omega^2 U_1 + \omega^4 U_2 + \dots \\ a(\alpha,\beta) &= A + \omega^2 A_1 + \omega^4 A_2 + \dots \end{aligned}\right\} \tag{17a}$$

$$\left.\begin{aligned} x(\alpha,\beta) &= X + \omega^2 X_1 + \omega^4 X_2 + \dots \\ t(\alpha,\beta) &= T + \omega^2 T_1 + \omega^4 T_2 + \dots \end{aligned}\right\} \tag{17b}$$

substituting into equations (7) and (8), and setting the coefficient of each power of ω equal to zero, the following sets of equations are obtained:

$$\left.\begin{aligned} U_\alpha + \lambda A_\alpha &= 0 \\ U_\beta - \lambda A_\beta &= 0 \end{aligned}\right\} \tag{18a}$$

$$\left.\begin{aligned} X_\alpha - (U + A)T_\alpha &= 0 \\ X_\beta - (U - A)T_\beta &= 0 \end{aligned}\right\} \tag{18b}$$

$$\left.\begin{aligned} U_{1\alpha} + \lambda A_{1\alpha} &= XT_\alpha \\ U_{1\beta} - \lambda A_{1\beta} &= XT_\beta \end{aligned}\right\} \tag{19a}$$

$$\left.\begin{aligned} X_{1\alpha} - (U + A)T_{1\alpha} &= (U_1 + A_1)T_\alpha \\ X_{1\beta} - (U - A)T_{1\beta} &= (U_1 - A_1)T_\beta \end{aligned}\right\} \tag{19b}$$

$$\left.\begin{aligned} U_{2\alpha} + \lambda A_{2\alpha} &= X_1 T_\alpha + XT_{1\alpha} \\ U_{2\beta} - \lambda A_{2\beta} &= X_1 T_\beta + XT_{1\beta} \end{aligned}\right\} \tag{20a}$$

$$\left.\begin{aligned} X_{2\alpha} - (U + A)T_{2\alpha} &= (U_1 + A_1)T_{1\alpha} + (U_2 + A_2)T_\alpha \\ X_{2\beta} - (U - A)T_{2\beta} &= (U_1 - A_1)T_{1\beta} + (U_2 - A_2)T_\beta \end{aligned}\right\} \tag{20b}$$

and so forth. The development and use of these equations is referred to as "method B."

Equations (18) and equations (13) and (14) are exactly the same and their solution is the complete solution of the problem when $\omega = 0$. This solution, the basic solution, is essential to the solution of the higher approximations and is considered known. The equations for the higher approximations in both methods are all linear in the unknowns.

Application to Specific Boundary Conditions

In order to demonstrate some of the differences between the two methods, an application to a simple case for which the equations can be integrated analytically is now made. Assume that the tube extends from $x = x_0$ to $x = x_1$ $(x_1 > x_0)$ and is closed at the ends. The tube is initially at rest, and the state of the gas is given by $u = 0$ and $a = a_0$, a constant. At time $t = 0$ the tube is started rotating about $x = 0$ with angular velocity ω. The neglect of cross flows and lateral pressure gradients might be completely unjustified in this case; nevertheless, the problem is interesting mathematically.

The problem, then, is to solve the primary equations (eqs. (5)) with the following boundary conditions:

At $t = 0$,

$$u = 0; \quad a = a_0 \qquad (x_0 \leqq x \leqq x_1) \tag{21a}$$

At $x = x_0$ and $x = x_1$,

$$u = 0 \qquad (t \geqq 0) \tag{21b}$$

The basic solution (i.e., for $\omega = 0$) is, of course,

$$\left.\begin{aligned} U &= 0 \\ A &= a_0 \end{aligned}\right\} \tag{22}$$

Since these are constants, this problem is similar to that of ordinary linearized theory for steady supersonic flow about slender bodies.

The introduction of α and β is somewhat arbitrary as long as equations (13) are satisfied; these parameters may be conveniently taken so that the two families of characteristics of the basic solution are given by

$$\left.\begin{aligned} x &= a_0 t + \beta \\ x &= -a_0 t + \alpha \end{aligned}\right\} \tag{23}$$

so that

$$\left.\begin{aligned} x &= \frac{\alpha + \beta}{2} \\ t &= \frac{\alpha - \beta}{2a_0} \end{aligned}\right\} \tag{24}$$

For method B, x and t are replaced by X and T in equations (23) and (24).

When ω has a finite value, the xt-plane may be divided into regions by the characteristics which pass through the points $(x_0,0)$ and $(x_1,0)$ as shown in figure 2. The solution in region I is unaffected by the walls at the ends. In region II the effect of the wall at $x = x_0$ only is felt; whereas, in region III that at $x = x_1$ only is felt. Both walls affect the flow in region IV.

The solution in region I is found first. (In appendix B the solution in region I is found in closed form.) With method A, equations (22) and (24) are substituted into equations (15). The first equation of equations (15) is integrated with respect to α, the second with respect to β, so that

$$\left.\begin{aligned} u_1 + \lambda a_1 &= \frac{1}{8a_0}\left[\alpha^2 + 2\alpha\beta + f(\beta)\right] \\ u_1 - \lambda a_1 &= -\frac{1}{8a_0}\left[2\alpha\beta + \beta^2 + g(\alpha)\right] \end{aligned}\right\} \tag{25}$$

where $f(\beta)$ and $g(\alpha)$ are arbitrary functions subject to the boundary conditions. These functions are determined by noting that at $t = 0$, $\alpha = \beta$, by equations (23), and that $u_1 = a_1 = 0$ by equations (21a). It is found that $f(\beta) = -3\beta^2$ and $g(\alpha) = -3\alpha^2$. By substituting these expressions into equations (25) and solving for u_1 and a_1, the following expressions are obtained:

$$\left.\begin{aligned} u_1 &= \frac{1}{a_0}\left(\frac{\alpha + \beta}{2}\right)\left(\frac{\alpha - \beta}{2}\right) = xt \\ a_1 &= -\frac{1}{2\lambda a_0}\left(\frac{\alpha - \beta}{2}\right)^2 = -\frac{a_0}{2\lambda}t^2 \end{aligned}\right\} \tag{26}$$

Equations (16) can now be solved in the same manner. Thus,

$$\left.\begin{aligned} u_2 &= -\frac{1}{3a_0^3}\left(\frac{\alpha + \beta}{2}\right)\left(\frac{\alpha - \beta}{2}\right)^3 = -\frac{1}{3}xt^3 \\ a_2 &= \frac{2\lambda + 3}{24\lambda^2 a_0^3}\left(\frac{\alpha - \beta}{2}\right)^4 = \frac{(2\lambda + 3)a_0}{24\lambda^2}t^4 \end{aligned}\right\} \tag{27}$$

Substituting these expressions into equations (9) gives as the solution in region I

$$\left.\begin{aligned} u &= \omega^2 xt - \frac{1}{3}\omega^4 xt^3 + \ldots \\ \frac{a}{a_0} &= 1 - \frac{1}{2\lambda}\omega^2 t^2 + \frac{2\lambda + 3}{24\lambda^2}\omega^4 t^4 + \ldots \end{aligned}\right\} \tag{28}$$

The solution by method B starts with equations (19a), which are the same as equations (15) in this case. The solution, then, is

$$\left.\begin{aligned} U_1 &= \frac{1}{a_0}\left(\frac{\alpha + \beta}{2}\right)\left(\frac{\alpha - \beta}{2}\right) = XT \\ A_1 &= -\frac{1}{2\lambda a_0}\left(\frac{\alpha - \beta}{2}\right)^2 = -\frac{a_0}{2\lambda}T^2 \end{aligned}\right\} \tag{29a}$$

Now that these values are known, equations (19b) may be solved. The arbitrary functions are determined by making lines in this solution pass through the same points on the x-axis of the xt-plane as the corresponding lines of the basic solution; that is, $X_1 = T_1 = 0$ when $\alpha = \beta$. Thus,

$$\left.\begin{aligned} X_1 &= \frac{1}{2a_0{}^2}\left(\frac{\alpha + \beta}{2}\right)\left(\frac{\alpha - \beta}{2}\right)^2 = \frac{1}{2}\,XT^2 \\ T_1 &= \frac{\lambda + 1}{6\lambda a_0{}^3}\left(\frac{\alpha - \beta}{2}\right)^3 = \frac{\lambda + 1}{6\lambda}\,T^3 \end{aligned}\right\} \tag{29b}$$

Solving equations (20) in the same way gives

$$\left.\begin{aligned} U_2 &= \frac{2\lambda + 1}{6\lambda a_0{}^3}\left(\frac{\alpha + \beta}{2}\right)\left(\frac{\alpha - \beta}{2}\right)^3 = \frac{2\lambda + 1}{6\lambda}\,XT^3 \\ A_2 &= -\frac{2\lambda + 1}{24\lambda^2 a_0{}^3}\left(\frac{\alpha - \beta}{2}\right)^4 = -\frac{2\lambda + 1}{24\lambda^2}\,a_0 T^4 \end{aligned}\right\} \tag{30a}$$

$$\left.\begin{aligned} X_2 &= \frac{5\lambda + 4}{24\lambda a_0{}^4}\left(\frac{\alpha + \beta}{2}\right)\left(\frac{\alpha - \beta}{2}\right)^4 = \frac{5\lambda + 4}{24\lambda}\,XT^4 \\ T_2 &= \frac{(\lambda + 1)(5\lambda + 7)}{120\lambda^2 a_0{}^5}\left(\frac{a - \beta}{2}\right)^5 = \frac{(\lambda + 1)(5\lambda + 7)}{120\lambda^2}\,T^5 \end{aligned}\right\} \tag{30b}$$

Substitution into the series (17) gives as the solution for region I

$$\left.\begin{aligned} u &= \omega^2 XT + \frac{2\lambda + 1}{6\lambda}\,\omega^4 XT^3 + \ldots \\ \frac{a}{a_0} &= 1 - \frac{1}{2\lambda}\,\omega^2 T^2 - \frac{2\lambda + 1}{24\lambda^2}\,\omega^4 T^4 + \ldots \end{aligned}\right\} \tag{31a}$$

$$\left.\begin{aligned} x &= X + \frac{1}{2}\,\omega^2 X T^2 + \frac{5\lambda + 4}{24\lambda}\,\omega^4 X T^4 + \ldots \\ t &= T + \frac{\lambda + 1}{6\lambda}\,\omega^2 T^3 + \frac{(\lambda + 1)(5\lambda + 7)}{120\lambda^2}\,\omega^4 T^5 + \ldots \end{aligned}\right\} \tag{31b}$$

The variables X and T may be interpreted simply as parameters of the solution. Actually, they define points of intersection of characteristics in the basic solution corresponding to characteristics which intersect at x,t in this solution. In any case, X and T may be eliminated from equations (31); the procedure results in equations (28) again.

The only difference in the solution for region II is in the determination of the arbitrary functions resulting from integration. Those functions resulting from integration with respect to β may be determined by the condition that the solution must equal that for region I on the boundary between the solutions; that is, when $\beta = x_0$. The other functions can then be found from the boundary condition given by equations (21b). That condition, however, does not apply to the solution of equations (19b) and (20b). For these equations, the boundary condition is replaced simply by

$$x = X + \omega^2 X_1 + \omega^4 X_2 + \ldots = x_0$$

or

$$X = x_0; \quad X_1 = X_2 = \ldots = 0$$

The solution found by method A is

$$\left.\begin{aligned}
u &= \frac{\omega^2}{a_0}(x - x_0)(a_0t + x_0) - \frac{\omega^4}{24\lambda a_0{}^3}(x - x_0)\Big\{8\lambda a_0{}^3t^3 + \\
&\quad 3(3\lambda - 7)x_0a_0{}^2t^2 - 12\Big[(\lambda + 4)x_0 - (\lambda + 2)x\Big]x_0a_0t + \\
&\quad \Big[(17\lambda + 31)x_0 - (5\lambda + 7)x\Big](x - x_0)x_0\Big\} + \dots \\
\frac{a}{a_0} &= 1 - \frac{\omega^2}{2\lambda a_0{}^2}\Big[a_0{}^2t^2 + 2x_0a_0t - 2x_0(x - x_0)\Big] + \\
&\quad \frac{\omega^4}{24\lambda^2a_0{}^4}\Big\{(2\lambda + 3)a_0{}^4t^4 + (3\lambda + 5)x_0a_0{}^3t^3 + 12(x - x_0)x_0a_0{}^2t^2 + \\
&\quad 3\Big[5(\lambda + 3)x_0 - (5\lambda + 7)x\Big](x - x_0)x_0a_0t - \\
&\quad 4(\lambda + 2)(4x_0 - x)(x - x_0)^2x_0\Big\} + \dots
\end{aligned}\right\} \quad (32)$$

and by method B

$$\left.\begin{aligned}
u &= \frac{\omega^2}{a_0}\Big[Xa_0T - x_0(a_0T - X + x_0)\Big] + \frac{\omega^4}{24\lambda a_0^{3}}\Bigg(4(2\lambda + 1)Xa_0^{3}T^3 - \\
&\quad x_0(a_0T - X + x_0)\Big\{4(2\lambda + 1)a_0^{2}T^2 - (13\lambda + 11)(X - x_0)a_0T - \\
&\quad (X - x_0)\Big[(\lambda - 1)X + (11\lambda + 13)x_0\Big]\Big\}\Bigg) + \dots \\
\frac{a}{a_0} &= 1 - \frac{\omega^2}{2\lambda a_0^{2}}\Big[a_0^{2}T^2 + 2x_0(a_0T - X + x_0)\Big] - \frac{\omega^2}{24\lambda^2 a_0^{4}}\Bigg((2\lambda + 1)a_0^{4}T^4 + \\
&\quad x_0(a_0T - X + x_0)\Big\{(7\lambda + 5)a_0^{2}T^2 + \Big[(11\lambda + 13)x_0 - (5\lambda + 7)X\Big]a_0T - \\
&\quad 2(X - x_0)\Big[(5\lambda + 7)x_0 - 2(\lambda + 2)X\Big]\Big\}\Bigg) + \dots
\end{aligned}\right\} \quad (33a)$$

$$\left.\begin{aligned}
x &= X + \frac{\omega^2}{2\lambda a_0^{2}}\Big[\lambda Xa_0^{2}T^2 - x_0(a_0T - X + x_0)(\lambda a_0T + X - x_0)\Big] + \dots \\
t &= T + \frac{(\lambda + 1)\omega^2}{12\lambda a_0^{3}}\Big[2a_0^{3}T^3 + 3x_0(a_0T - X + x_0)(a_0T + X - x_0)\Big] + \dots
\end{aligned}\right\} \quad (33b)$$

The solutions for region III are obtained in a similar way and are the same except that x_0 is replaced by x_1 and the signs of some terms are reversed. In regions II and III, it is found that the elimination of X and T from the solution by method B results again in the solution by method A.

Figure 3 shows characteristic diagrams for the case $x_1 = 2x_0$, $\omega = \frac{1}{2}\frac{a_0}{x_0}$, and $\lambda = 5$ $(\lambda = 1.4)$. The dashed lines are the characteristics of the basic solution and, therefore, are the mathematical characteristics used in the higher-order solutions; these lines are in the xt-plane for method A and in the XT-plane for method B. The solid lines are the true characteristic lines of the problem in the xt-plane. They are found automatically by method B. Method A, as described so far, does not give the true characteristic lines, but it is found that the solution of equations (7), after the substitution of the values of u and a (for example, eqs. (28) for region I), results in the same characteristic lines as have been found by method B. This determination is necessary for the evaluation of the flow in the field, as illustrated in figure 4 where typical distributions of velocity and speed of sound are shown for a given time and in figure 5 where typical variations with time at a given value of x are shown. In figures 4(a) and 5(a), the results found by method A without the determination of the variation of the characteristics are shown. That is, the lines PR and PS of figure 3 are used as the boundaries between the regions. It is seen that u and a are discontinuous at these boundaries (although the quantity $u - \lambda a$ is continuous along PR and $u + \lambda a$ is continuous along PS). In figures 4(b) and 5(b), the results are shown when the true characteristics of the problem, the lines QR and QS of figure 3, are taken as the boundaries. Method B automatically gives the results of figures 4(b) and 5(b). Parts (a) and (b) are identical except where they represent conditions in the regions between PR and QR and between PS and QS in figure 3.

DISCUSSION

Some aspects of the preceding example which are due to the fact that U and A are constants and which would not ordinarily be present should be noted. (1) The unsolved equations of method A are unusually simple in comparison with those of method B since the terms containing the unknowns explicitly drop out. (2) As a result, the solution of the equations of the first approximation is the same by both methods. (3) The solution in method B for X_1 and T_1 affects the unknowns u and a only in the second order and similarly for the higher approximations.

The basic solution cannot usually be found analytically but must be determined by a step-by-step procedure. In that case, the approximations must also be done step by step. The characteristic equations would be written in a finite difference form with $\Delta\alpha$ and $\Delta\beta$ cancelled out. Thus, α and β would not be used explicitly. With method B, the calculations would be carried out in the XT-plane where the characteristic directions remain constant as they do in the xt-plane in method A. The transfer to the xt-plane, which consists of merely adding the terms of equations (17b), can be delayed until the desired number of approximations has been made.

From the preceding simple example, some inferences can be drawn concerning the application of the two methods to other basic equations. The unknowns do not occur explicitly (although they do appear as derivatives) in equations (19a) and (20a), but, if they occur thus in the basic characteristic equations, corresponding to equations (18a), they will occur in the equations corresponding to (19a) and (20a). If the original independent variables (X and T in the given example) appear in the basic characteristic equations, the four equations in each approximation of method B must all be solved simultaneously, rather than as two pairs of equations as in the example. This complication is a serious disadvantage of method B and may rule it out completely in such cases.

The relative simplicity of the equations of method B, taken one by one, is probably quite general. The appearance of derivatives of lower approximations with respect to the original independent variables in the equations of method A is to be expected as long as the primary equations are nonlinear, whereas they cannot appear in method B.

If an application is made to the two-dimensional supersonic flow about a single body, for example, and conditions on the surface of the body only are desired, the calculation of the true characteristics is superfluous. In cases such as this, method A has the advantage of having two less equations to solve in each approximation. However, the preceding problem indicates that the true position of characteristics through points of discontinuity in the boundary conditions is necessary for the determination of the flow in the field and for the determination of conditions on another boundary if such characteristics intersect it.

This necessity for correcting characteristics as it applies to linearized theory for steady supersonic flow is discussed in reference 6. There, as in the present example, what is here called the basic solution consists of constants so that this effect is felt only in the second and higher approximations. In more general cases it would be felt in the first approximation.

In problems where the perturbation involves a change in the position of the boundary, method B has the advantage that the position is constant in the plane of the original independent variables of the basic equations (for example, the XT-plane in the present problem) where the solution is carried out.

When the perturbation involves the addition of a new dependent variable, it can be handled in much the same way as the entropy is handled in this case (appendix A).

CONCLUDING REMARKS

Two methods of using the concept of linearized characteristics are derived for the one-dimensional unsteady flow in a rotated tube. A comparison of both methods has shown that they give the same results but there are significant differences in their application.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., July 11, 1952.

APPENDIX A

DEVELOPMENT OF EQUATIONS FOR VARIABLE ENTROPY

When the entropy is not constant throughout the fluid, equations (5) are replaced by

$$\left.\begin{aligned} &\lambda a_t + a u_x + \lambda u a_x = 0 \\ &u_t + u u_x + \lambda a a_x - \frac{1}{\gamma} a^2 s_x - \omega^2 x = 0 \end{aligned}\right\} \tag{34}$$

where s is the entropy increment divided by the gas constant. As long as the flow is without shocks, the entropy obeys the relation

$$s_t + u s_x = 0 \tag{35}$$

The characteristic equations become

$$\left.\begin{aligned} &u_a + \lambda a_\alpha - \frac{1}{\gamma} a s_\alpha - \omega^2 x t_\alpha = 0 \\ &u_\beta - \lambda a_\beta + \frac{1}{\gamma} a s_\beta - \omega^2 x t_\beta = 0 \\ &x_\alpha = (u + a) t_\alpha \\ &x_\beta = (u - a) t_\beta \end{aligned}\right\} \tag{36}$$

and equation (35) may be replaced by

$$s_\alpha t_\beta + s_\beta t_\alpha = 0 \tag{37}$$

(See ref. 3 for a derivation of the preceding equations for $\omega = 0$.)

Writing

$$s(x,t) = S + \omega^2 s_1 + \omega^4 s_2 + \ldots$$

substituting this expression along with equations (9) into equations (34) and (35), and converting to characteristic form give the equations of method A as

$$\left.\begin{aligned} &U_\alpha + \lambda A_\alpha - \frac{1}{\gamma} AS_\alpha = 0 \\ &U_\beta - \lambda A_\beta + \frac{1}{\gamma} AS_\beta = 0 \\ &S_\alpha t_\beta + S_\beta t_\alpha = 0 \end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned} &u_{1\alpha} + \lambda a_{1\alpha} + \left(U_x + \lambda A_x\right) t_\alpha \left(u_1 + a_1\right) - \frac{1}{\gamma}\left[As_{1\alpha} + \left(2a_1 + u_1\right) S_\alpha\right] - xt_\alpha = 0 \\ &u_{1\beta} - \lambda a_{1\beta} + \left(U_x - \lambda A_x\right) t_\beta \left(u_1 - a_1\right) + \frac{1}{\gamma}\left[As_{1\beta} + \left(2a_1 - u_1\right) S_\beta\right] - xt_\beta = 0 \\ &s_{1\alpha} t_\beta + s_{1\beta} t_\alpha + 2u_1 S_x t_\alpha t_\beta = 0 \end{aligned}\right\} \tag{39}$$

$$\left.\begin{aligned} &u_{2\alpha} + \lambda a_{2\alpha} + \left(U_x + \lambda A_x\right) t_\alpha \left(u_2 + a_2\right) + \left(u_{1x} + \lambda a_{1x}\right) t_\alpha \left(u_1 + a_1\right) - \\ &\quad \frac{1}{\gamma}\left[As_{2\alpha} + \left(2a_1 + u_1\right) s_{1\alpha} + \left(2a_2 + u_2\right) S_\alpha + \left(u_1 + a_1\right)^2 S_x t_\alpha\right] = 0 \\ &U_{2\beta} - \lambda a_{2\beta} + \left(U_x - \lambda A_x\right) t_\beta \left(u_2 - a_2\right) + \left(u_{1x} - \lambda a_{1x}\right) t_\beta \left(u_1 - a_1\right) + \\ &\quad \frac{1}{\gamma}\left[As_{2\beta} + \left(2a_1 - u_1\right) s_{1\beta} + \left(2a_2 - u_2\right) S_\beta - \left(u_1 - a_1\right)^2 S_x t_\beta\right] = 0 \\ &s_{2\alpha} t_\beta + s_{2\beta} t_\alpha + 2\left(u_1 s_{1x} + u_2 S_x\right) t_\alpha t_\beta = 0 \end{aligned}\right\} \tag{40}$$

and so forth. Equations (13) still give the mathematical characteristic direction for each approximation. If the true characteristics are to be found, equations (7) still apply.

In order to obtain the equations for method B, the expression

$$s(\alpha,\beta) = S + \omega^2 S_1 + \omega^4 S_2 + \ldots$$

along with equations (17) is substituted into equations (36) and (37), so that

$$\left.\begin{aligned}
&U_\alpha + \lambda A_\alpha - \frac{1}{\gamma} AS_\alpha = 0\\
&U_\beta - \lambda A_\beta + \frac{1}{\gamma} AS_\beta = 0\\
&X_\alpha - (U + A)T_\alpha = 0\\
&X_\beta - (U - A)T_\beta = 0\\
&S_\alpha T_\beta + S_\beta T_\alpha = 0
\end{aligned}\right\} \tag{41}$$

$$\left.\begin{aligned}
&U_{1\alpha} + \lambda A_{1\alpha} - \frac{1}{\gamma}\left(A_1 S_\alpha + AS_{1\alpha}\right) = XT_A\\
&U_{1\beta} - \lambda A_{1\beta} + \frac{1}{\gamma}\left(A_1 S_\beta + AS_{1\beta}\right) = XT_\beta\\
&X_{1\alpha} - (U + A)T_{1\alpha} = \left(U_1 + A_1\right)T_\alpha\\
&X_{1\beta} - (U - A)T_{1\beta} = \left(U_1 - A_1\right)T_\beta\\
&S_{1\alpha}T_\beta + S_\alpha T_{1\beta} + S_{1\beta}T_\alpha + S_\beta T_{1\alpha} = 0
\end{aligned}\right\} \tag{42}$$

$$\left.\begin{aligned}
U_{2\alpha} + \lambda A_{2\alpha} - \frac{1}{\gamma}\left(A_2 S_\alpha + A_1 S_{1\alpha} + A S_{2\alpha}\right) &= X_1 T_\alpha + X T_{1\alpha} \\
U_{2\beta} - \lambda A_{2\beta} + \frac{1}{\gamma}\left(A_2 S_\beta + A_1 S_{1\beta} + A S_{2\beta}\right) &= X_1 T_\beta + X T_{1\beta} \\
X_{2\alpha} - (U + A) T_{2\alpha} &= \left(U_1 + A_1\right) T_{1\alpha} + \left(U_2 + A_2\right) T_\alpha \\
X_{2\beta} - (U - A) T_{2\beta} &= \left(U_1 - A_1\right) T_{1\beta} + \left(U_2 - A_2\right) T_\beta \\
S_{2\alpha} T_\beta + S_{1\alpha} T_{1\beta} + S_\alpha T_{2\beta} + S_{2\beta} T_\alpha + S_{1\beta} T_{1\alpha} + S_\beta T_{2\alpha} &= 0
\end{aligned}\right\} \tag{43}$$

and so forth.

It is to be noted that, when the entropy of the primary solution S is a constant, the equations of both methods simplify considerably. For each approximation in this case, the equations for the entropy may be solved first, then the equations for the velocity and speed of sound, and then (in method B) the characteristic direction equations. Otherwise, all three equations (in method A) or all five equations (in method B) in each approximation must be solved simultaneously.

APPENDIX B

SOLUTION IN CLOSED FORM FOR REGION I

The form of equations (28) suggests the substitutions

$$u = xf(t)$$

$$a = a_0 g(t)$$

Equations (5) become

$$\lambda \frac{dg}{dt} + gf = 0 \tag{44}$$

$$\frac{df}{dt} + f^2 - \omega^2 = 0 \tag{45}$$

whereas the boundary condition given by equations (21a) becomes

$$f(0) = 0; \quad g(0) = 1$$

Equation (45) is easily integrated to give

$$f = \omega \tanh \omega t$$

By substituting this value for f, equation (44) can be solved to give

$$g = (\cosh \omega t)^{-1/\lambda}$$

Therefore

$$u = \omega x \tanh \omega t$$

$$\frac{a}{a_0} = (\cosh \omega t)^{-1/\lambda}$$

These functions may be expanded in power series; the procedure results in equations (28) once more.

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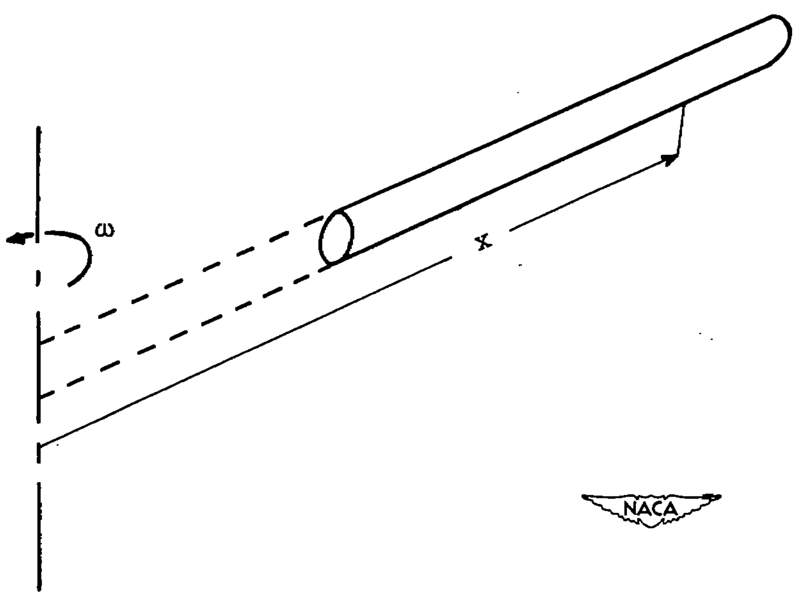


Figure 1.- The rotating tube.

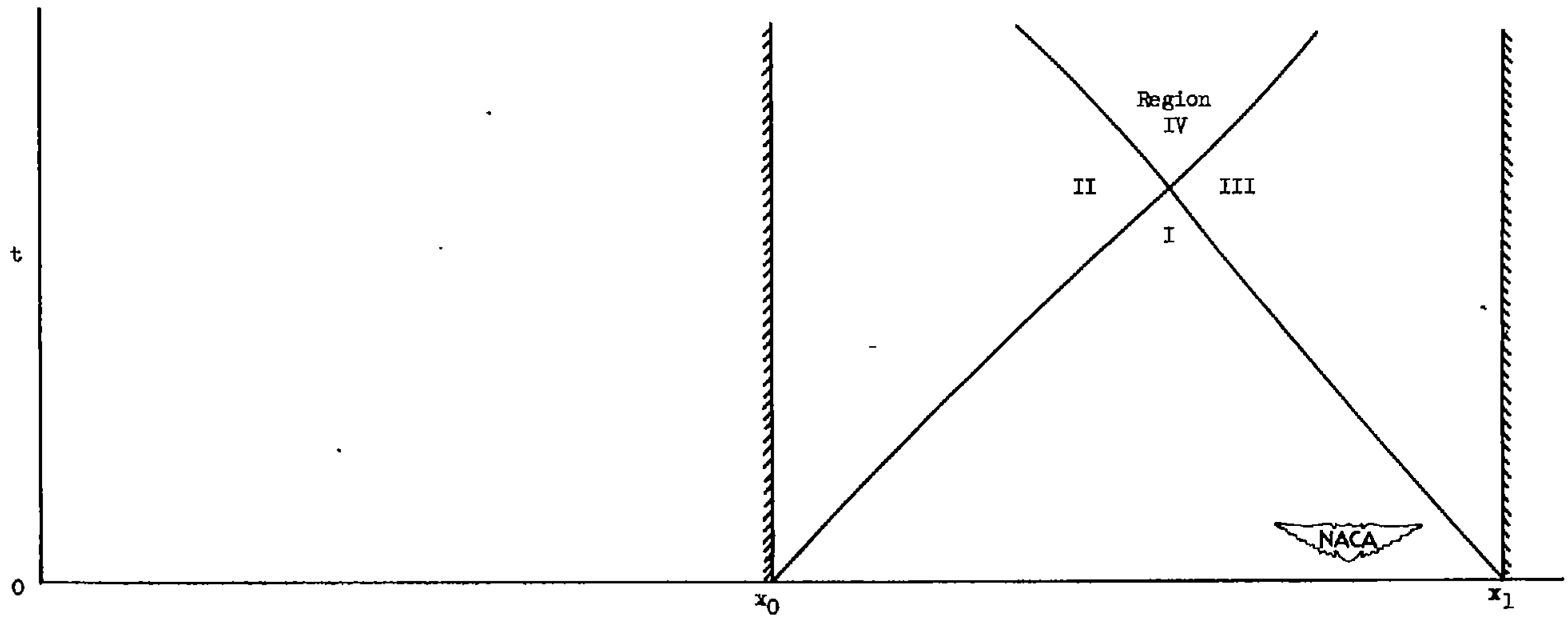


Figure 2.- Regions in the xt-plane influenced by the walls.

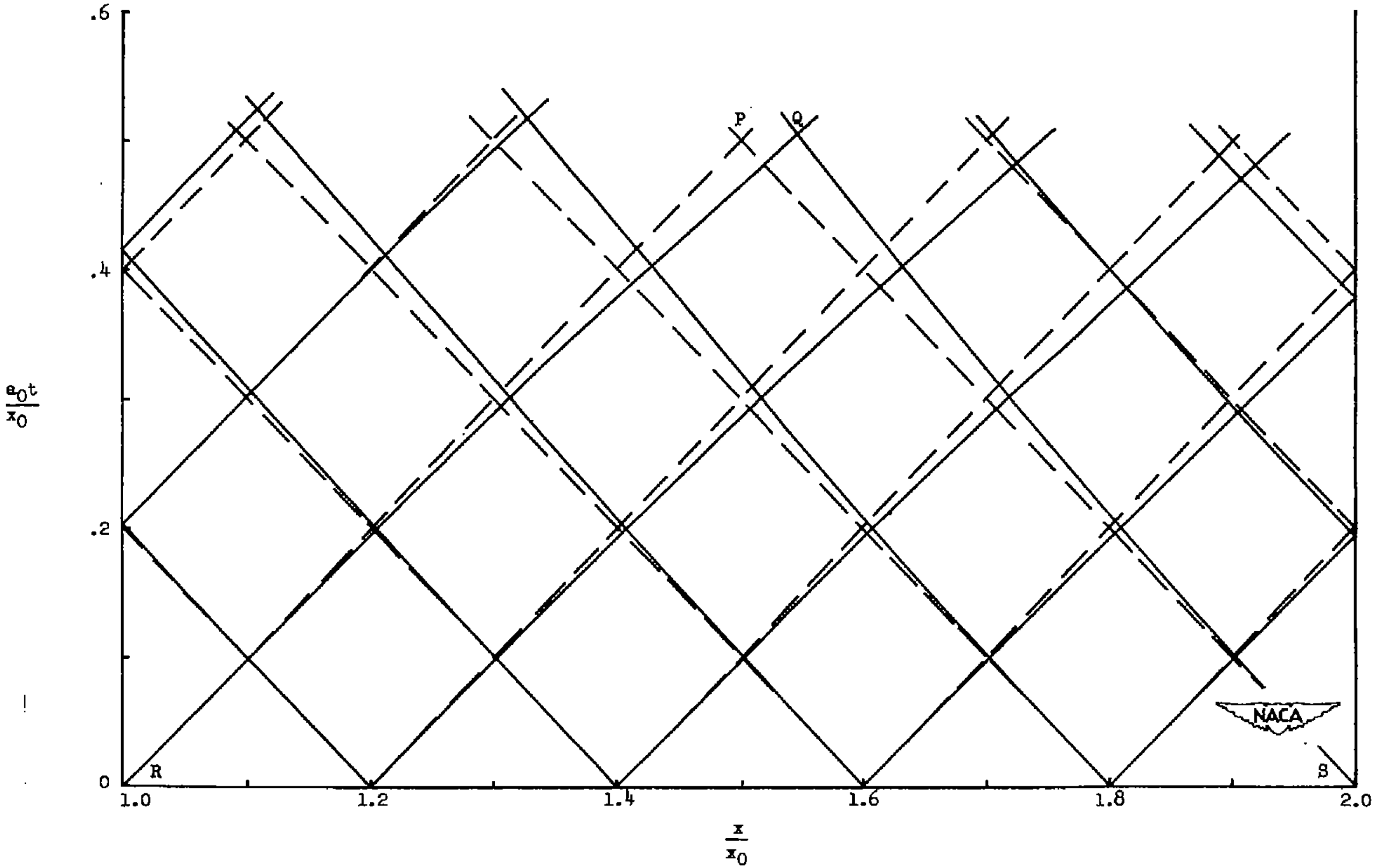
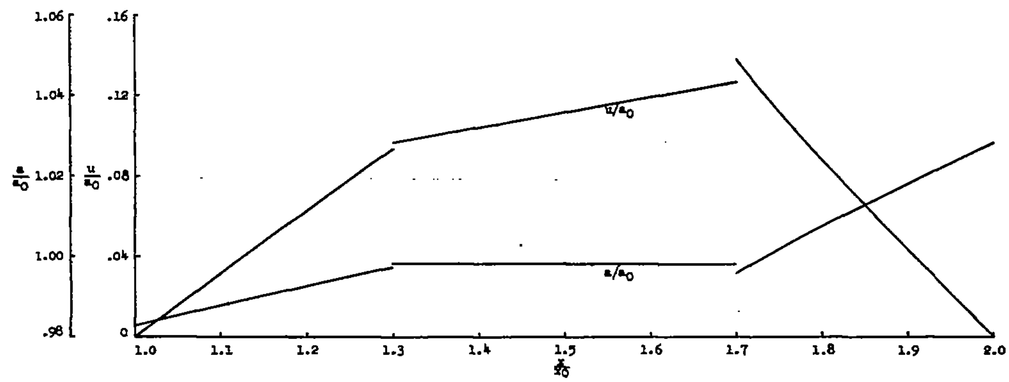
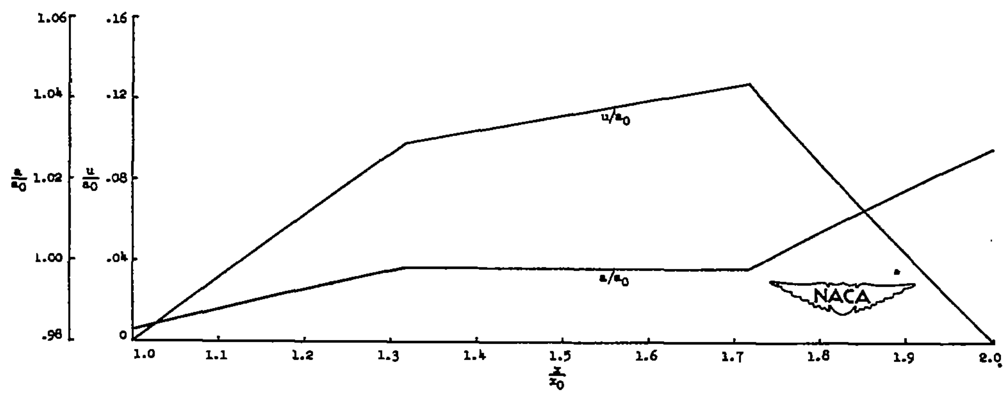


Figure 3.- Characteristic lines for the case $x_1 = 2x_0$, $\omega = \frac{1}{2}\frac{a_0}{x_0}$, and $\lambda = 5$ ($\gamma = 1.4$).

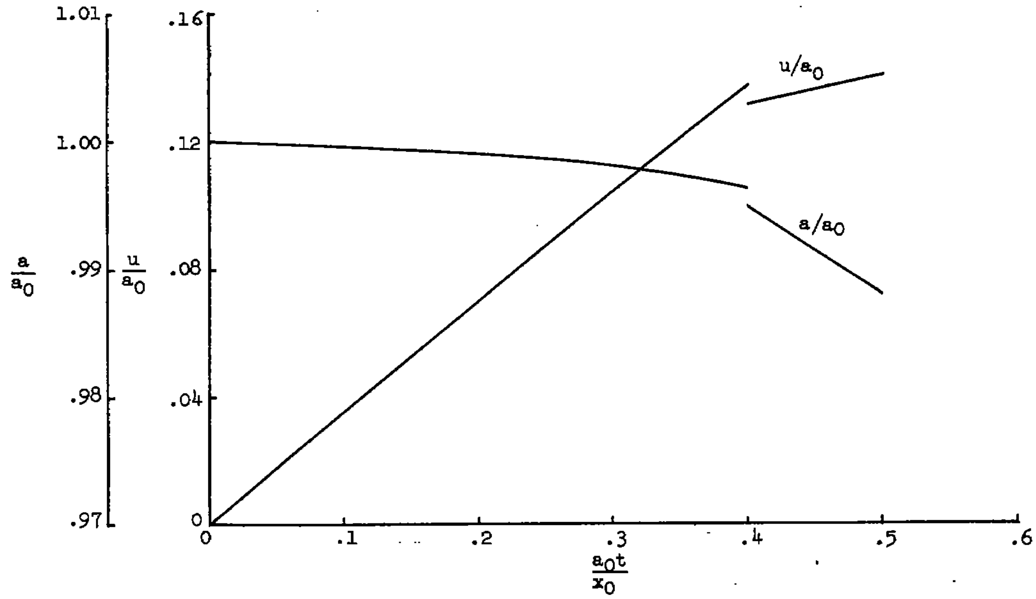


(a) With mathematical characteristics of method A as boundaries between regions.

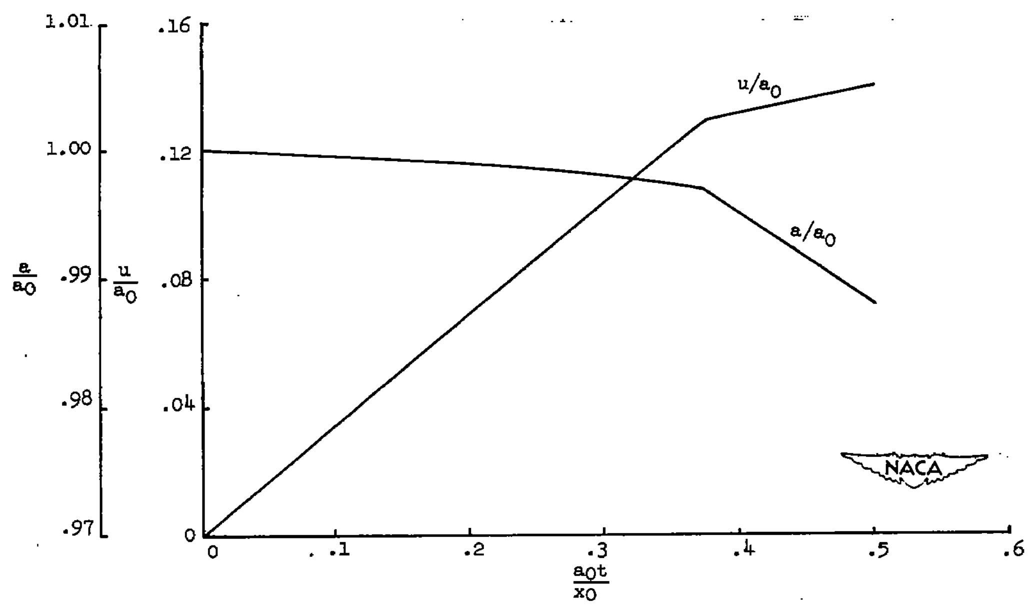


(b) With true characteristics as boundaries between regions.

Figure 4.- Distribution of velocity and speed of sound when $\frac{a_0 t}{x_0} = 0.3$ for the case $x_1 = 2x_0$, $\omega = \frac{1}{2}\frac{a_0}{x_0}$, and $\lambda = 5$ $(\gamma = 1.4)$.



(a) With mathematical characteristics of method A as boundaries between regions.



(b) With true characteristics as boundaries between regions.

Figure 5.- Variation of velocity and speed of sound with time at $\frac{x}{x_0} = 0.4$ for the case $\omega = \frac{1}{2} \frac{a_0}{x_0}$ and $\lambda = 5$ $(\gamma = 1.4)$.

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